Privacy-Preserving Approximation of Transcendental Functions

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Big questions

- Cryptosystems allow us to **store and transmit** data securely.
  - To send a message securely, encrypt it into a secure form, which can then be decrypted by an authorized party.
- Can **computation** be performed securely?
  - If so, what kinds of computation are possible?

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Can we perform secure computation *efficiently*?

- To some extent, using **partially homomorphic encryption (PHE)**.
  - PHE only allows *some* operations to be performed securely.
  - PHE schemes are significantly less time-intensive than current FHE schemes.

The Paillier cryptosystem

The **Paillier cryptosystem** was published by Pascal Paillier in 1999, in the paper *Public-Key Cryptosystems Based on Composite Degree Residuosity Classes*.

Limitations of the Paillier cryptosystem:

1. Only allows for the encryption and manipulation of **integers**
2. A **partially homomorphic cryptosystem**, i.e. supports limited secure computation
   - A. Given two encrypted integers, can obtain encryption of their sum.
   - B. Given an encrypted integer and an unencrypted integer, can obtain encryption of their sum.
   - C. Given an encrypted integer and an unencrypted integer, can obtain encryption of their product.

Extensions to the Paillier cryptosystem

- There is a known method to extend Paillier to **floating-point numbers**, allowing secure addition/subtraction and secure plaintext multiplication.
- In 2017, Boukoros, Karvelas, and Katzenbeisser presented a protocol to perform secure **division** in a two-party system.
- In 2009, Erkin, et al. presented a protocol to perform secure **squaring** in a two-party system.
- We can use the squaring protocol to perform secure **multiplication** in a two-party system.
  - Suppose we have the encryptions of a and b.
  - We can compute the encryptions of $(a+b)^2$, a^2, and b^2.
  - Then, compute the encryption of $\frac{1}{2}((a+b)^2 - a^2 - b^2) = ab$.

What we know so far

Using the Paillier cryptosystem and known extensions, we are able to

- Add and subtract encrypted numbers;
- Multiply and divide encrypted numbers;
- Raise an encrypted number to an integer power.

But can we do more?

What we know so far

Using the Paillier cryptosystem and known extensions, we are able to

- ▷ Add and subtract encrypted numbers;
- ▷ Multiply and divide encrypted numbers;
- ▷ Raise an encrypted number to an integer power.

But can we do more? **Yes**.

- ▷ Compute logarithms of encrypted numbers
- ▷ Compute the inverse tangent of encrypted numbers

Conventional methods for logarithm approximation

Remarks on using conventional methods to approximate the logarithm of an encrypted number:

- Series approximation — the Maclaurin series representation of $\log(1+x)$ only converges for $-1 \leq x \leq 1$.
- Lookup tables — cannot be done securely as we cannot expose the value of the encrypted number.

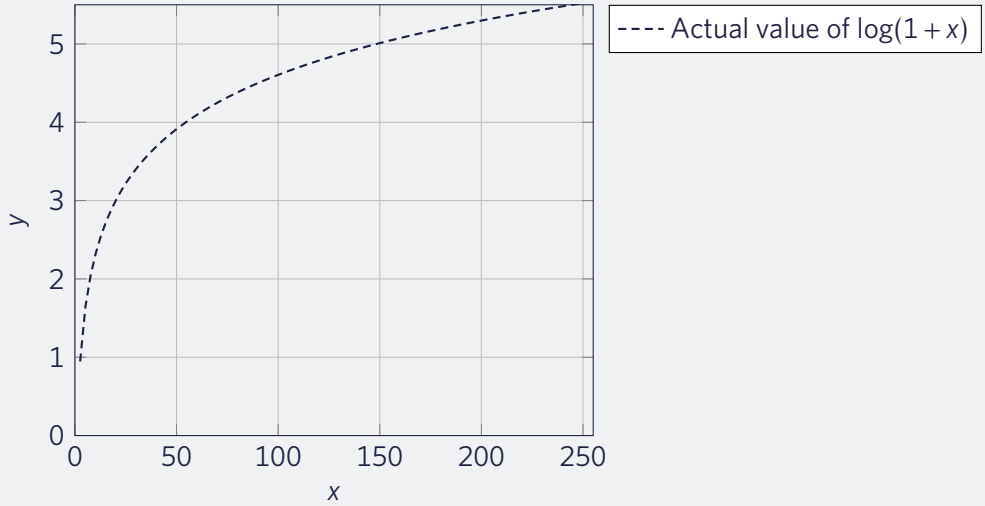
Known closed-form approximation to the logarithm

Khattri presented the following approximation for $\log(1+x)$:

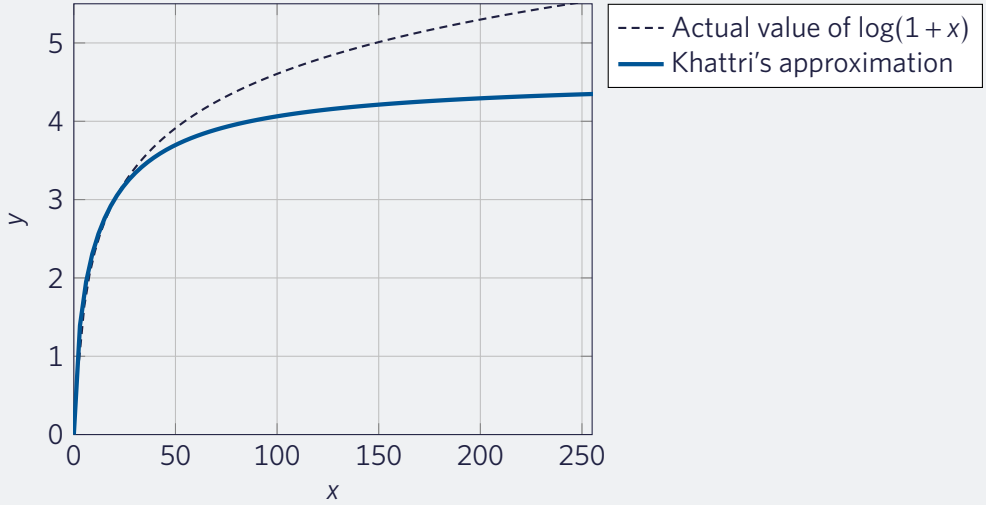
$$\log(1+x) \approx \frac{137x^5 + 2310x^4 + 9870x^3 + 15120x^2 + 7560x}{30x^5 + 900x^4 + 6300x^3 + 16800x^2 + 18900x + 7560}.$$

This was derived by approximating the integral $\int_n^{n+1} \frac{1}{t}\,\mathrm{d}t = \log\left(1 + \frac{1}{n}\right)$ using 5-point Gauss–Legendre quadrature.

Known closed-form approximation to the logarithm



Known closed-form approximation to the logarithm



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$$\log(1+x) \approx \frac{a(x)}{b(x)} + \log 16,$$

where

$$a(x) = 137x^5 + 26685x^4 + 617370x^3 - 6498630x^2 - 121239315x - 257804775$$
$$b(x) = 30(x^5 + 405x^4 + 27210x^3 + 488810x^2 + 2536005x + 3122577).$$

Using Python 3.6.5, we implemented this approximation to compute the logarithm of numbers encrypted under the Paillier cryptosystem.

New closed-form approximation to the logarithm

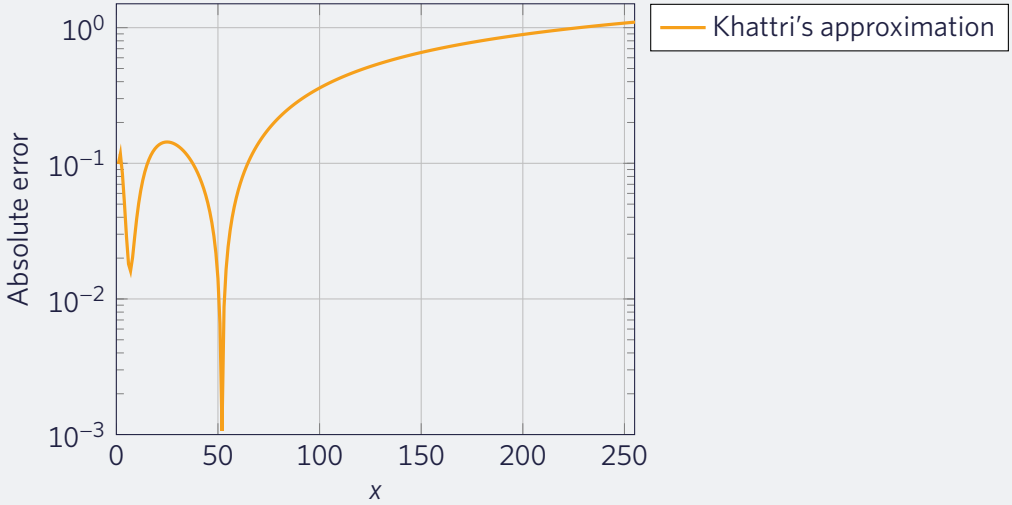
We derived this approximation by approximating the integral

$$\log\left(1+\frac{1}{n}\right) = \int_{n}^{\alpha n+\alpha} \frac{1}{t}\,\mathrm{d}t - \log\alpha,$$

where $\alpha = 1/16$, using 5-point Gauss-Legendre quadrature.

It can be proven numerically that choosing $\alpha = 1/16$ minimizes the absolute error of the approximation from $\log(1+x)$ in the range $x \in [0, 255]$.

Comparison of logarithm approximations



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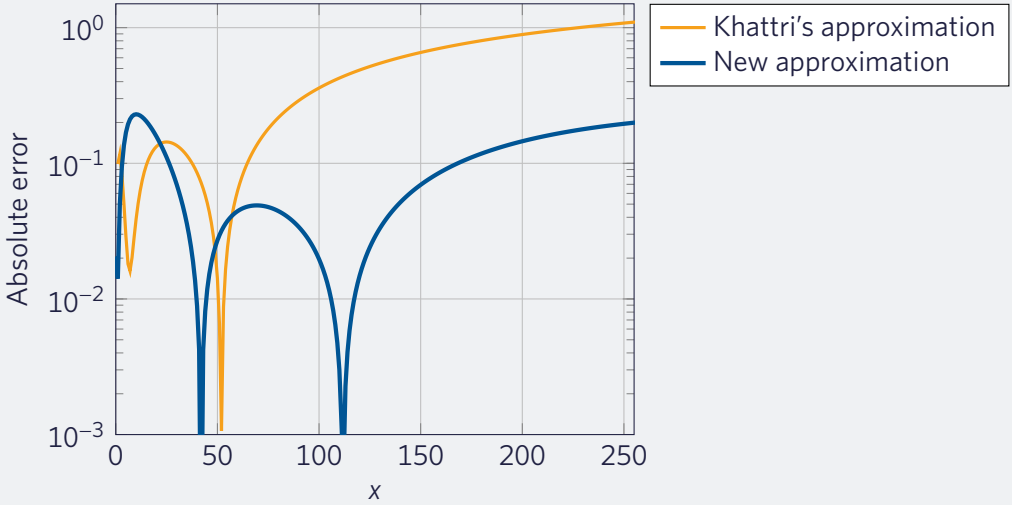

Comparison of logarithm approximations

Table 1: Comparison of logarithm approximation errors under Paillier for selected inputs

Input	Khattri's approximation	New approximation
$\log 1$	0	2.053729×10^{-2}
$\log 10$	2.833179×10^{-2}	2.280124×10^{-1}
$\log 50$	2.170779×10^{-2}	2.437148×10^{-2}
$\log 100$	3.524916×10^{-1}	2.122159×10^{-2}
$\log 200$	8.859869×10^{-1}	1.441611×10^{-1}
$\log 256$	1.098463×10^{0}	1.993136×10^{-1}

Conventional methods for inverse tangent approximation

Remarks on using conventional methods to approximate the inverse tangent of an encrypted number:

- Series approximation — there are two major series representations for $\arctan x$
  - The Maclaurin series expansion for $\arctan x$ only converges for $-1 \leq x \leq 1$.
  - In 1755, Euler discovered the following series for the inverse tangent,

$$\arctan x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}},$$

    which converges for all values of x.
- Lookup tables — cannot be done securely as we cannot expose the value of the encrypted number.

New approximation for inverse tangent

We now present the approximation:

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$$\arctan x \approx \frac{4\left(225x^9 + 15925x^7 + 144753x^5 + 350595x^3 + 238140x\right)}{15\left(x^{10} + 480x^8 + 11760x^6 + 64120x^4 + 114660x^2 + 63504\right)}.$$

We derived this approximation by approximating the integral $\arctan x = \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t$, using 5-point Gauss–Legendre quadrature.

New approximation for inverse tangent

We compared our new approximation to a partial sum consisting of the first five terms of Euler's series, as it requires similar number of floating-point operations to the new approximation.

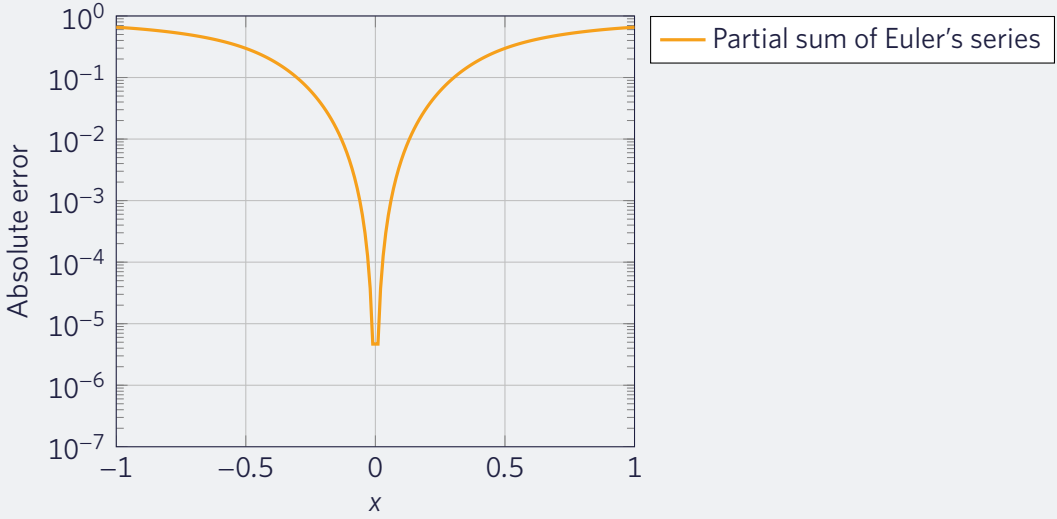
$$\sum_{n=0}^{4} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}} = \frac{965x^9 + 2370x^7 + 2688x^5 + 1470x^3 + 315x}{315\left(x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1\right)}$$

(Euler's series)

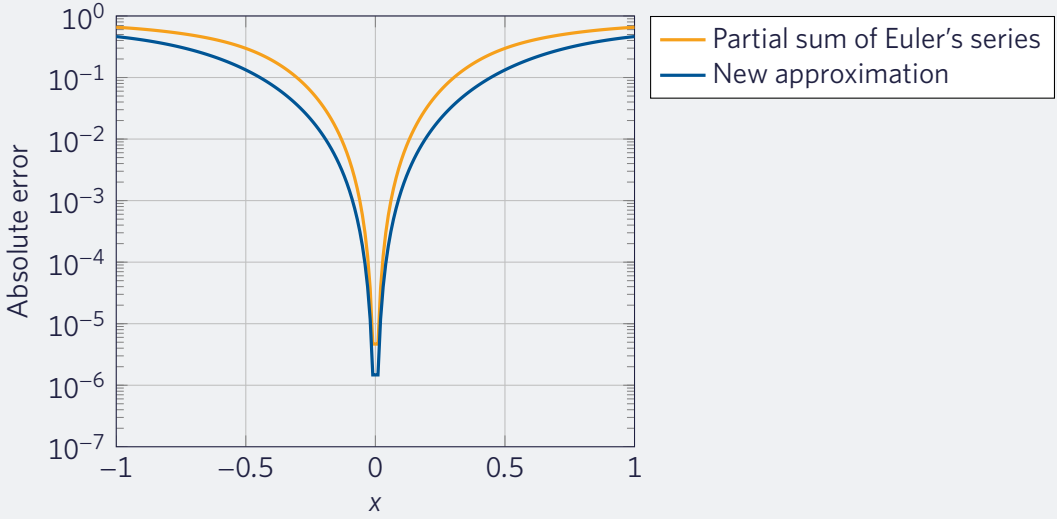
$$\arctan x \approx \frac{4\left(225x^9 + 15925x^7 + 144753x^5 + 350595x^3 + 238140x\right)}{15\left(x^{10} + 480x^8 + 11760x^6 + 64120x^4 + 114660x^2 + 63504\right)}.$$

(New approximation)

Comparison of inverse tangent approximations



Comparison of inverse tangent approximations



Comparison of inverse tangent approximations

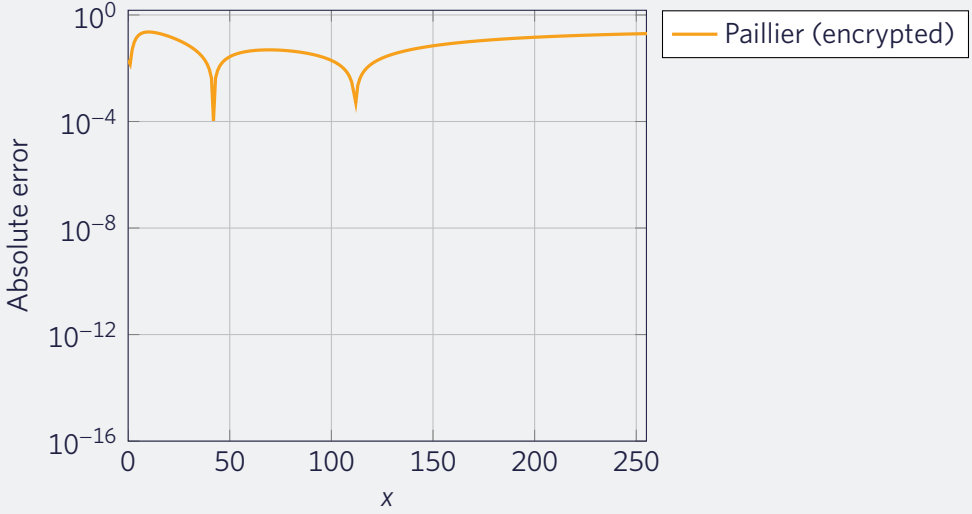
Table 2: Comparison of inverse tangent approximation errors under Paillier for selected inputs

Input	Partial sum of Euler's series	New approximation
$\arctan -1$	6.564089×10^{-1}	4.629557×10^{-1}
$\arctan -0.1$	4.523405×10^{-3}	1.451929×10^{-3}
$\arctan -0.01$	4.665260×10^{-6}	1.472027×10^{-6}
$\arctan 0$	0	0
$\arctan 0.01$	4.665228×10^{-6}	1.472020×10^{-6}
$\arctan 0.1$	4.523405×10^{-3}	1.451929×10^{-3}
$\arctan 1$	6.564089×10^{-1}	4.629557×10^{-1}

Remarks on new approximations

- Numerical integration can be used when infinite series representations can't be used.
- Approximation for $\log(1+x)$ was tailored for input range $0 \leq x \leq 255$. Different ranges can be accommodated by tweaking the approximation procedure.
- Approximation for $\arctan x$ was tested for the input range $-1 \leq x \leq 1$. Inputs outside this range can be accommodated using trigonometric identities.
- Noise was observed when the approximations were implemented using the Paillier cryptosystem.
  - This is likely due to errors in floating-point representation.
  - Approximations are significantly more accurate when evaluated using SageMath, i.e. unencrypted.

Experimental accuracy of new log approximation



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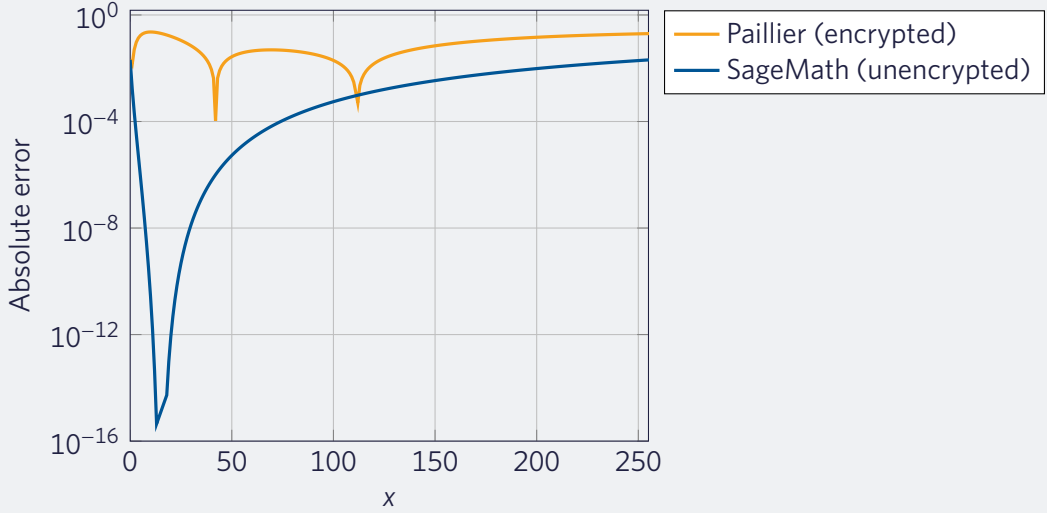
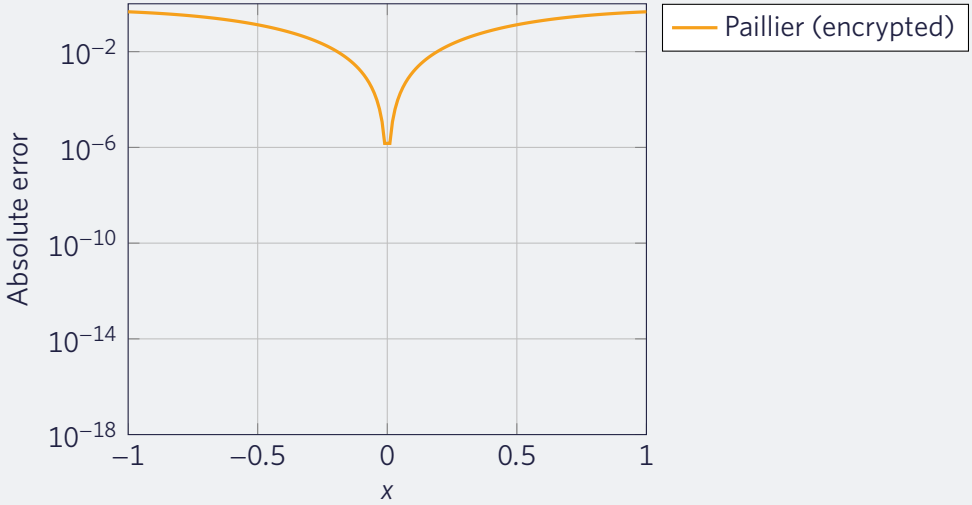


Experimental accuracy of new inverse tangent approximation



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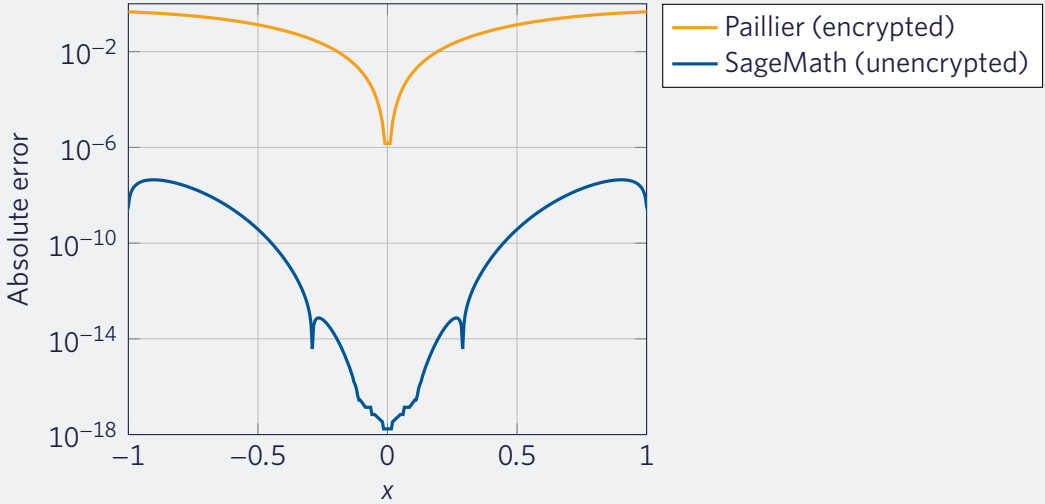


Conclusion and recommendations

- We have shown new closed-form approximations to the logarithm and inverse tangent functions.
- We have demonstrated how these can be used with the Paillier cryptosystem to allow relatively accurate privacy-preserving transcendental function calculation.
- For further research:
  - Explore other numerical integration methods
  - Explore other homomorphic cryptosystems

Thank you!

Gauss–Legendre quadrature

We first convert the integral to an integral over the interval $[-1, 1]$ using the following transformation:

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)\mathrm{d}x.$$

Gauss–Legendre quadrature

Then, we approximate the integral using the following summation:

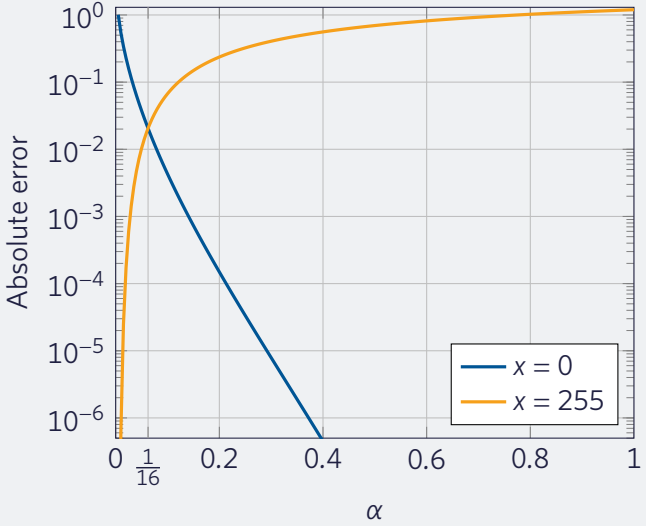
$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{5} w_i f(x_i),$$

where

$$w_1 = 0,$$
$$w_2 = \frac{1}{21}\sqrt{245 - 14\sqrt{70}},$$
$$w_3 = -\frac{1}{21}\sqrt{245 - 14\sqrt{70}},$$
$$w_4 = \frac{1}{21}\sqrt{245 + 14\sqrt{70}},$$
$$w_5 = -\frac{1}{21}\sqrt{245 + 14\sqrt{70}},$$

$$x_1 = \frac{128}{225},$$
$$x_2 = \frac{1}{900}\left(322 + 13\sqrt{70}\right),$$
$$x_3 = \frac{1}{900}\left(322 + 13\sqrt{70}\right),$$
$$x_4 = \frac{1}{900}\left(322 - 13\sqrt{70}\right),$$
$$x_5 = \frac{1}{900}\left(322 - 13\sqrt{70}\right).$$

Optimizing the logarithm approximation



Optimizing the logarithm approximation